Counting Factors

Charlie wants to know how many factors 360 has.
How would you work it out?

Charlie started by working out the prime factorisation of 360.

$$
\begin{aligned}
360 &= 2 \times 180 \\
&= 2 \times 2 \times 90 \\
&= 2 \times 2 \times 2 \times 45 \\
&= 2 \times 2 \times 2 \times 3 \times 15 \\
&= 2 \times 2 \times 2 \times 3 \times 3 \times 5
\end{aligned}
$$

So $360 = 2^3 \times \times 3^2 \times 5$.

Then he made a table to list the 24 possible combinations of the prime factors.

<table>
<tr><td colspan="6">2^0</td><td colspan="6">2^1</td><td>…</td><td colspan="3">2^3</td></tr>
<tr><td colspan="2">3^0</td><td colspan="2">3^1</td><td colspan="2">3^2</td><td colspan="2">3^0</td><td colspan="2">3^1</td><td colspan="2">3^2</td><td colspan="2">…</td><td colspan="2">3^2</td></tr>
<tr><td>5^0</td><td>5^1</td><td>5^0</td><td>5^1</td><td>5^0</td><td>5^1</td><td>5^0</td><td>5^1</td><td>5^0</td><td>5^1</td><td>5^0</td><td>5^1</td><td colspan="2">…</td><td>5^0</td><td>5^1</td></tr>
</table>

So the first branch gives us $2^0 \times 3^0 \times 5^0 = 1$
the second branch gives us $2^0 \times 3^0 \times 5^1 = 5$ …
… the fourth branch gives us $2^0 \times 3^1 \times 5^1 = 15$ …
… the eleventh branch gives us $2^1 \times 3^2 \times 5^0 = 18$, and so on.

Charlie thinks these numbers also have exactly 24 factors. Can you use Charlie's method to explain why?

$$25725 = 5^2 \times 3^1 \times 7^3$$

$$217503 = 11^1 \times 13^3 \times 3^2$$

$$312500 = 5^7 \times 2^2$$

$$690625 = 17^1 \times 13^1 \times 5^5$$

$$94143178827 = 3^{23}$$

Can you find a number with exactly 14 factors?
Can you find the smallest such number?

Which numbers have an odd number of factors?